MID-SEMESTER EXAMINATION
M. MATH II YEAR, II SEMESTER 2014-2014
ERGODIC THEORY

Max. Score:100 Time limit: 3hrs.

The 7 questions below carry a total of 110 marks. Answer as many questions as you can. The maximum you can score is 100.

1. Prove that $\frac{a_1(\omega)+a_2(\omega)+\ldots+a_n(\omega)}{n} \to \infty$ for almost all $\omega \in (0,1)$ where $a_n(\omega) = [\frac{1}{T^{n-1}(\omega)}]$ and T is the Gauss transformation.
Hint: use a truncation argument. [15]

2. Using the Ergodic Theorem prove that $\frac{\#\{n \leq N : a_n(\omega)=1\}}{N} \to \frac{1}{2}$ for almost all $\omega \in (0,1)$ where $a_n(\omega)$ is the $n-th$ coefficient in the expansion of ω to base 2. [10]
Hint: consider the transformation $T\omega = 2\omega \,\mathrm{mod}(1)$.

3. Which of the following transformations are uniquely ergodic? Justify.

a) $T : \mathbb{R} \to \mathbb{R}, T(x) = x + 1$ ($\mathcal{F}$ =Borel sigma field)

b) $\Omega = \{1, 2, 3\}, T1 = 2, T2 = 3, T3 = 1$ ($\mathcal{F}$ =Power set)

c) $\Omega = \{1, 2, 3\}, T1 = 1, T2 = 3, T3 = 2$ ($\mathcal{F}$ =Power set) [5+5+5]

4. Consider a Markov chain with transition matrix $\begin{pmatrix} 2/3 & 0 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$.
Let T be the Markov shift corresponding to this Markov chain. Is T ergodic? Is it weak-mixing? Is it strong-mixing? [15]
[You may use any theorem proved in class]

5. Let $Tz = az, z \in S^1$ where $a \in S^1$ is not a root of unity. Show that the only eigen values of T (i.e. eigen values of the corresponding operator on $L^2(S^1)$) are the numbers $a^n, n \in \mathbb{Z}$. [20]
Hint: use Fourier series expansions of eigen functions.

6. Prove the following theorem:

Let $(\Omega, \mathcal{F}, P, T)$ be a dynamical system and $\mathcal{G}_n = \{T^{-n}(A) : A \in \mathcal{F}\}, \mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_n$.
If every set in $\mathcal{G}$ has probability 0 or 1 then T is strong mixing. [20]

7. Prove Poincare recurrence Theorem: $\{P(A) > 0, P \circ T^{-1} = P\} \Rightarrow P\{\omega \in A : T^n\omega \notin A$ for all n sufficiently large$\} = 0$. [15]